Strong convergence of unitary and permutation representations of discrete groups

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Abstract

We survey a research program on the strong convergence of unitary and permutation representations of discrete groups. We also take the opportunity to flesh out details that have not appeared elsewhere.

Contents

1 Definitions

Throughout the article Γ denotes an infinite discrete group. For $n \in \mathbb{N}$ we write $\mathcal{U}(n)$ for the group of complex $n \times n$ unitary matrices and S_n for the group of permutations of $[n] \stackrel{\text{def}}{=} \{1, \ldots, n\}$.

Definition 1.1 (Strong convergence, P**Mat**F)**.** If $\{\rho_i : \Gamma \to \mathcal{U}(n_i)\}_{i=1}^{\infty}$ are a sequence of (possibly random) finite dimensional unitary representations of Γ, say ρ_i strongly converge to the regular representation (almost surely, or in probability, if ρ_i are random) if for any $z \in \mathbb{C}[\Gamma]$,

$$\lim_{i\to\infty} \|\rho_i(z)\| = \|\lambda_\Gamma(z)\|$$

(a.s., or in probability, respectively) where $\lambda_\Gamma : \Gamma \to U(\ell^2(\Gamma))$ is the left regular representation. The norms above are operator norms. We write $\rho_i \xrightarrow{\text{strong}} \lambda_\Gamma$ in this event. If Γ has such a sequence of unitary representations then we say Γ is purely matricial field (P**Mat**F).

Remark 1.2.

1. Weak convergence. Some authors ask for weak convergence as part of the definition of strong convergence, but we do not. Weak convergence is the statement that for all $z \in \mathbb{C}[\Gamma]$,

   $$\lim_{i\to\infty} \mathrm{tr}[\rho_i(z)] = \tau(z)$$

   where $\mathrm{tr} = n_i^{-1}\mathrm{Tr}$ is the normalized trace on $n_i \times n_i$ matrices and $\tau(z)$ is the canonical tracial state on the reduced C^*-algebra $C^*_{\mathrm{red}}(\Gamma)$. If $C^*_{\mathrm{red}}(\Gamma)$ has a unique tracial state then strong convergence implies weak convergence (e.g. [LM23, Lemma 6.1])[1]. We now know exactly when $C^*_{\mathrm{red}}(\Gamma)$ has a unique tracial state by Breuillard-Kalantar-Kennedy-Ozawa [BKKO17, Thm. 1.6]: this is so when Γ has no non-trivial normal amenable subgroup.

2. Nomenclature. Let M_{n_i}denote the complex matrix C^*-algebra of dimension n_i, $\ell^\infty(\prod_i M_{n_i})$ denote the bounded sequences in the product, and $\mathcal{I}$ the closed two sided ideal of sequences that converge to zero. If $\rho_i \xrightarrow{\text{strong}} \lambda_\Gamma$ then $\prod_i \rho_i$ descends (and extends) to an embedding

   $$C^*_{\mathrm{red}}(\Gamma) \hookrightarrow \ell^\infty(\prod_i M_{n_i})/\mathcal{I}.$$

   If there exists any such embedding then $C^*_{\mathrm{red}}(\Gamma)$ is called matricial field by Blackadar and Kirchberg in [BK97]. But conversely such an

[1]We first heard this observation from Benoît Collins in June 2022.

embedding does not (a priori) have to factor through $\ell^\infty(\prod_i M_{n_i})$ when restricted to $\mathbb{C}[\Gamma]$. Hence the adjective 'purely'. The concept of purely matricial field was introduced in Magee–de la Salle [MdlS23].

3. More generally, if G is a locally compact topological group, we can extend our definition by replacing $\mathbb{C}[\Gamma]$ by the continuous compactly supported complex functions on G.

A first attempt to require strong convergence to factor through permutations would be that the ρ_i are the composition of some $\phi_i \in \mathrm{Hom}(\Gamma, S_n)$ with the (n-dimensional) permutation representations of S_n. However, since this gives rise to ρ_i with non-zero invariant vectors, this can never work if Γ is non-amenable, which is not satisfactory. So the definition is modified in the following way. Let std denote the $(n-1)$-dimensional irreducible subrepresentation of the defining representation of S_n.

Definition 1.3 (P**Perm**F)**.** If there exist a sequence of homomorphisms $\{\phi_i : \Gamma \to S_{n_i}\}_{i=1}^\infty$ such that

$$\{\rho_i \stackrel{\text{def}}{=} \mathrm{std} \circ \phi_i\}_{i=1}^\infty$$

strongly converge to the regular representation, then we say Γ is purely permutation field (P**Perm**F).

Strong convergence, and the associated above properties P**Mat**F and P**Perm**F have recently found powerful applications in a surprising range of settings including but not limited to the spectral geometry of graphs [BC19] and hyperbolic manifolds [HM23, LM23, MT24], the theory of minimal surfaces (observed by Song [Son24a, Son24b], and the Peterson-Thom conjecture [PT11] (as observed by Hayes [Hay22]) on subalgebras of free group factors.

Acknowledgments

This paper is a survey of some of my results with collaborators together with things I thought about during a stay at I.A.S. during 2023-2024. I am extremely grateful to the I.A.S. for this opportunity. Some particular elements of this year have had a large effect on this document. The conversations I had with Mikael de la Salle who patiently explained many things to me. The seminar at Princeton University organized by Peter Sarnak on 'C^*-algebras and related topics' was extremely stimulating and sparked many conversations.

Indeed, I also learned many interesting things from subsequent conversations with Ramon van Handel and his lectures in the seminar.

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2 P**Mat**F and P**Perm**F

We begin with some basic properties of P**Mat**F and P**Perm**F. In this section $\Lambda \leqslant \Gamma$ are discrete groups.

Lemma 2.1 (Restriction to subgroups)**.** *If* Γ *is* P**Mat**F *(resp.* P**Perm**F*) then* Λ *is* P**Mat**F *(resp.* P**Perm**F*).*

Proof. Suppose $z \in \mathbb{C}[\Lambda] \leqslant \mathbb{C}[\Gamma]$. If $\rho_i \xrightarrow{\text{strong}} \lambda_\Gamma$ then $\|\rho_i(z)\| \to \|\lambda_\Gamma(z)\|$ as $i \to \infty$. But as a $\mathbb{C}[\Lambda]$-module,

$$\ell^2(\Gamma) \cong \bigoplus_{[\gamma] \in \Lambda \backslash \Gamma} \ell^2(\Lambda\gamma) \cong \bigoplus_{[\gamma] \in \Lambda \backslash \Gamma} \ell^2(\Gamma)$$

so $\|\lambda_\Gamma(z)\| = \|\lambda_\Lambda(z)\|$. Hence if ρ_i' is the restriction of ρ_i to Λ, $\rho_i' \xrightarrow{\text{strong}} \lambda_\Lambda$. If the representations factor through S_{n_i} then their restrictions still do. $\square$

Lemma 2.2 (Induction to finite index overgroups)**.** *If* Λ *is* P**Mat**F *and finite index in* Γ *then* Γ *is also* P**Mat**F*.*

Lemma 2.2 is a special case of a more general phenomenon (induction from co-compact lattices) that will be covered later in the paper (see Theorem 3.1).

2.1 Amenable groups.

The following argument about amenable groups was obtained in conversations with Mikael de la Salle.

Say that Γ is residually linear (RL) (resp. residually finite (RF)) if it embeds into a product of GL_{n_i} (resp. S_{n_i}). A theorem of Malcev [Mal40] states that every finitely generated (f.g.) linear group is residually finite. Hence if Γ is f.g. and RL then it is RF.

If Γ is not RL then there is $\gamma \in \Gamma$ that is killed by every homomorphism to some GL_n. Then considering

$$z = \gamma - \mathrm{id} \in \mathbb{C}[\Gamma],$$

we have

$$\|\rho(z)\| = 0$$

for every $\rho : \Gamma \to \mathrm{GL}_n$. But $\|\lambda(z)\|$ is not zero (or else $\lambda(z) = 0$ but $\mathbb{C}[\Gamma]$ always embeds to $C^*_{\mathrm{red}}(\Gamma)$). The upshot is that:

Lemma 2.3. *A* P**Mat**F *group is residually linear. Hence it is also either residually finite or not finitely generated.*

On the other hand, if Γ is countable and RF then by projecting to large enough factors of the $\prod_i S_{n_i}$ we obtain a sequence of homomorphisms $\phi_j : \Gamma \to S_{N_j}$ such that each ϕ_j injects on a finite set S_j such that $\cup_j S_j = \Gamma$. For any $z \in \mathbb{C}[\Gamma]$ let Φ_j denote the unitary representation of Γ obtained by composing ϕ_j with the standard representation of S_{N_j}.

Because we eventually inject on the support of $(zz^*)^p$ we have then

$$\mathrm{tr}[\Phi_j(zz^*)^p] = \tau((zz^*)^p) - \frac{1}{N_j}\epsilon((zz^*)^p)$$

where ϵ is the state associated to the trivial representation of Γ, for $j \gg_p 1$. Since the trivial representation is weakly contained in the regular representation of Γ we have

$$\epsilon((zz^*)^p) \leqslant \|\lambda(z)\|^{2p}.$$

Hence we also have

$$\|\Phi_j(z)\|^{2p} = \|\Phi_j(zz^*)^p\| \geqslant \mathrm{tr}[\Phi_j(zz^*)^p] \to_{i\to\infty} \tau((zz^*)^p) = (\|\lambda(z)\| + o_{p\to\infty}(1))^{2p}$$

where the last equality used that the trace τ is faithful. Taking p large and fixed and letting $i \to \infty$ for each large p we obtain

$$\liminf_{j\to\infty} \|\Phi_j(z)\| \geqslant \|\lambda(z)\|.$$

But on the other hand for amenable groups $\|\lambda(z)\| \geqslant \|\Phi_i(z)\|$ for all $z \in \mathbb{C}[\Gamma]$ because all finite dimensional unitary representations of amenable groups are weakly contained in the regular representation. So in fact

$$\lim_{j\to\infty} \|\Phi_j(z)\| = \|\lambda(z)\|.$$

Hence in summary for amenable groups

$$\text{countable} + \text{RF} \implies \text{P}\mathbf{Perm}\text{F}$$

and

$$\text{P}\mathbf{Mat}\text{F} \implies \text{RL},$$

$$\text{f.g.} + \text{P}\mathbf{Mat}\text{F} \implies \text{RF}.$$

What is perhaps surprising is that all amenable groups have f.d. *approximate representations* (in norm sense) that strongly converge to the regular representation [TWW17]. This appears to leave open the question of whether non-countable discrete amenable groups are P**Mat**F or P**Perm**F.

2.2 Free groups

Let $\mathbf{F}_r$ denote a free non-abelian group of rank $r \geqslant 2$.

The original interest in property P**Mat**F for non-amenable groups comes from an observation of Voiculescu [Voi93] that establishing '$\mathbf{F}_r$ is P**Mat**F' — albeit not in this language — would settle the then outstanding problem in operator algebras to prove the K-theoretic construct $\text{Ext}(C^*_{\text{red}}(\mathbf{F}_r))$ has non-invertible elements.

Motivated by this application to K-theory, Haagerup and Thorbjørnsen proved $\mathbf{F}_r$ is P**Mat**F in the main theorem of [HT05]. Importantly, this, and all subsequent proofs of this fact, rely on random matrix theory. As a result, there is a lacuna in the field: we do not know how to construct explicit f.d. representations of free groups that strongly converge to the regular representation[2] Haagerup and Thorbjørnsen used GUE random matrices and an application of functional calculus to prove their result, and as a result, it left open the question of whether Haar distributed random representations of $\mathbf{F}_r$ a.s. strongly converge to the regular representation. This was proved by Collins and Male in [CM14].

Later on, Bordenave and Collins [BC19] proved that $\mathbf{F}_r$ is P**Perm**F by use of random permutation representations. To explain their motivation and to explain the motivation for P**Perm**F in general, we detour to discuss Friedman's theorem.

[2] This problem seems known in the community, it is certainly not my own, and it was highlighted to me by Avi Wigderson.

Friedman's theorem [Fri08], formerly conjecture of Alon [Alo86], states informally that random regular graphs are almost optimal expanders. Alon did not state which random model to use, but besides, there are contiguity results that tell us it does not matter for the purposes of proving the above statement [Wor99]. For simplicity of exposition assume that the random regular graph has degree $2r$ and n vertices, with r fixed and $n \to \infty$.

If $\sigma_1, \ldots, \sigma_r$ are i.i.d. uniformly random elements of S_n then the Schreier graph for these generators and the action of S_n on $\{1, \ldots, n\}$ has adjacency matrix

$$2r \cdot \mathrm{id}_{\mathbb{C}} \oplus \left[\mathrm{std}(\sigma_1) + \mathrm{std}(\sigma_1^{-1}) + \cdots + \mathrm{std}(\sigma_r) + \mathrm{std}(\sigma_r^{-1})\right].$$

The $2r \cdot \mathrm{id}_{\mathbb{C}}$ factor corresponds to the trivial eigenvalue $2r$ and in this setting, asymptotically optimal expansion amounts to

$$\|\mathrm{std}(\sigma_1) + \mathrm{std}(\sigma_1^{-1}) + \cdots + \mathrm{std}(\sigma_r) + \mathrm{std}(\sigma_r^{-1})\| \to 2\sqrt{2r-1}$$

(in probability) where the right hand side is optimal by a result of Alon–Boppana [Nil91]. To put the above in a more symmetric form, and to connect to our other discussions, if

$$\phi(x_i) \stackrel{\mathrm{def}}{=} \sigma_i, \quad \rho(x_i) = \mathrm{std}(\phi(x_i)) = \mathrm{std}(\sigma_i)$$

then ρ_i is a random representation of the type appearing in the definition of P**Perm**F and Friedman's theorem states

$$\|\rho_i(x_1 + x_1^{-1} + \cdots + x_r + x_r^{-1})\| \to \|\lambda_{\mathbf{F}_r}(x_1 + x_1^{-1} + \cdots + x_r + x_r^{-1})\| = 2\sqrt{2r-1}$$

in probability. This is a very restricted version of P**Perm**F concerning only the linear element

$$z = x_1 + x_1^{-1} + \cdots + x_r + x_r^{-1} \in \mathbb{C}[\mathbf{F}_r]$$

whereas P**Mat**F and P**Perm**F ask for the same for all(!) elements of $\mathbb{C}[\Gamma]$. This is precisely what Bordenave–Collins prove in [BC19]: for all $z \in \mathbb{C}[\mathbf{F}_r]$

$$\|\rho_i(z)\| \to \|\lambda_{\mathbf{F}_r}(z)\|$$

in probability. We cannot leave this discussion without remarking that very recently, a beautiful and short proof of Friedman's theorem has been obtained by Chen, Garza-Vargas, Tropp, and van Handel, [CGVTvH24]. They also extend the proof without adding much length to give a new proof of Bordenave–Collins' theorem.

2.3 Non-free examples

Limit groups and surface groups. Motivated in part by applications to hyperbolic surfaces (see §3), in [LM23] Louder and I studied *limit groups:* finitely generated groups such that for every finite subset, there is a homomorphism to a free group that injects on that set.

Theorem 2.4 (Louder–Magee)**.** *Every limit group is* P**Perm**F*.*

The full strength of this result uses a result of Sela [Sel01, CG05] stating that every limit group embeds into an iterated sequence of (extensions of centralizers), beginning with a free group. This is combined with a quantitative version of a lemma of Baumslag [Bau62] and Haagerup's inequality [Haa79]. The following is one of the motivations.

Corollary 2.5. *The fundamental group of an orientable surface of genus at least two is* P**Perm**F*.*

While stated as a corollary, Sela's work is not needed here as a genus two surface group

$$\langle a, b, c, d \mid [a, b][c, d] \rangle$$

explicitly embeds to the extension of centralizers

$$\langle a, b, t \mid [t, [a, b]] \rangle$$

via $a \mapsto a, b \mapsto b, c \mapsto b^t, d \mapsto a^t$; the injectivity of this map uses normal form for amalgamated products. Higher genus surface groups embed into this example via covering spaces.

Right-angled Artin Groups and related examples. A finitely generated right-angled Artin group (RAAG) is generated by finitely many generators with only relations that some subset of the pairs of generators are commuting pairs. Besides interpolating between free non-abelian and free abelian groups, they turn out to be fairly universal in that many natural families of groups virtually embed into RAAGs. That is, they embed after passing to a finite index subgroup. This includes the following classes of groups.

1. Closed hyperbolic three manifold fundamental groups. (Bergeron–Wise [BW12] and Agol – [Ago13])

2. Non-compact finite volume three manifold fundamental groups[3]. (Wise – [Wis21, Thm. 14.29])

[3] I thank Jean-Pierre Mutanguha for making me aware of this.

3. Arithmetic 'standard type' hyperbolic n manifold groups with $n \geqslant 4$. (Bergeron–Haglund–Wise [BHW11])

4. Any Coxeter group. (Haglund–Wise [HW10])

5. Any one-relator group with torsion. (Wise – [Wis21, Cor. 18.1])

6. Any word-hyperbolic cubulated group. (Agol – [Ago13])

The above results also all rely on a result of Haglund and Wise [HW08] stating that fundamental groups of compact special cube complexes embed into RAAGs.

By Lemmas 2.1 and 2.2, P**Mat**F passes from RAAGs to those groups above. The 'master theorem' that RAAGs are indeed P**Mat**F was obtained in joint work with Thomas [MT24].

Theorem 2.6 (Magee–Thomas)**.** *Finitely generated RAAGs are* P**Mat**F*.*

It is not true however that in general RAAGs are P**Perm**F. The following proposition has not appeared elsewhere in print.

Proposition 2.7. $\mathbf{F_2} \times \mathbf{F_2} \times \mathbf{F_2}$ *is not* P**Perm**F*.*

Proof. Let G_i denote the copy of $\mathbf{F}_2$ embedded at the i^{th} factor, $i = 1, 2, 3$. Let $\{\rho_j\}_{j=1}^{\infty}$ be a putative sequence of permutation representations of $\mathbf{F_2} \times \mathbf{F_2} \times \mathbf{F_2}$ such that $\mathrm{std} \circ \rho_i$ strongly converges to the regular representation. We interpret ρ_j as a linear representation by composition with the defining representation. Let $H_{i,j} \stackrel{\text{def}}{=} \rho_j(G_i) \leqslant S_{n_j}$. Since $\mathbf{F}_2$ is non-amenable, each $H_{i,j}$ has at most one dimensional space of invariant vectors in the defining representation. This means that each $H_{i,j}$ must be a transitive subgroup of S_{n_j} (acting transitively on $[n_j]$) for $j \gg 1$. A theorem of finite group theory[4] states that two commuting transitive permutation groups (say J_1 and J_2 in S_N) only arise in the following way: there is some auxiliary group U with $|U| = N$, and injective morphisms $\iota_1 : J_1 \cong \lambda(U)$ and $\iota_2 : J_2 \cong \rho(U)$ where $\lambda(U)$ (resp. $\rho(U)$) is the permutation group induced by multiplication by U on its left (resp. right).

[4]Indeed, J_1 must act freely on $[N]$, since J_2 preserves the fixed points of any $j \in J_1$. This identifies J_1 with $[N]$ via $j \mapsto j.1$ and makes the action of J_1 into the left regular action. Now $J_2 \leqslant \mathrm{Perms}(J_1)$ is in the centralizer of the left regular action of J_1, which is the right regular action of J_1 [Hal18, Thm. 6.3.1]. But since J_2 is transitive, it must be isomorphic to J_1 and equal to the whole right regular action.

In particular, in the situation above, $H_{1,j}$ and $H_{2,j}$ can be identified with $\lambda(U_j)$ and $\rho(U_j)$ for some j. There are now different ways to conclude; we note that $H_{3,j} \cong U_j$ by the same observation but in the above model, it is U_j acting on itself by permutations and commuting with $\lambda(U_j)$ and $\rho(U_j)$. Only the identity in U_j can do this, so U_j is the trivial one element group. Then $\mathrm{std} \circ \rho_j$ is the 0-dimensional representation for $j \gg 1$. This is obviously a contradiction. $\square$

As a byproduct of the above proof, one sees that any putative sequence of permutation representations of $\mathbf{F_2} \times \mathbf{F_2}$ such that $\mathrm{std} \circ \rho_i$ strongly converges to the regular representation has to have quite a particular structure. However, there is a candidate that fits this structure.

Question 2.8. *For $n \in \mathbb{N}$ let θ_n and ϕ_n denote uniformly random permutation representations $\mathbf{F}_2 \to S_n$. Then*

$$\lambda \circ \theta_n : \mathbf{F}_2 \to S_{n!} = \mathbf{Perm}(S_n), \; \rho \circ \phi_n : \mathbf{F}_2 \to S_{n!} = \mathbf{Perm}(S_n)$$

are two commuting permutation representations of $\mathbf{F}_2$. Is it true that as $n \to \infty$,

$$\mathrm{std}_{n!} \circ [\lambda \circ \theta_n \times \rho \circ \phi_n] \xrightarrow{\text{strong}} \lambda_{\mathbf{F_2} \times \mathbf{F_2}}$$

in probability?

Notice that if the above is true it entails that as $n \to \infty$, $\lambda \circ \theta_n \xrightarrow{\text{strong}} \lambda_{\mathbf{F_2}}$ in probability. This is far from known. In fact it is not known whether

$$\|\lambda \circ \theta_n[x_1 + x_2 + x_1^{-1} + x_2^{-1}]\| \to 2\sqrt{3} \tag{2.1}$$

in probability; or the same with '2' replaced by any $d \geqslant 0$ and $\sqrt{3}$ replaced by $\sqrt{2d-1}$. This is even stronger than the also open question:

'Are most fixed degree Cayley graphs of S_n uniform expanders?'. ((2.1) suggests they are moreover almost optimal expanders)

We note here an important and perhaps relevant result of Kassabov [Kas07] giving the existence of fixed degree uniformly expanding Cayley graphs of S_n with $n \to \infty$. In this vein there is a very recent work of E. Cassidy [Cas23, Cas24] who proves the strong convergence in probability of

$$\rho_n : \mathbf{F}_r \to \mathcal{U}(N(n))$$

$$\rho_n(x_i) \stackrel{\text{def}}{=} \pi_\lambda(\sigma_i)$$

where σ_i are as above and π_λ is the irreducible representation of S_n corresponding to a Young diagram λ with $1 \leqslant |\lambda| \leqslant n^{\frac{1}{13}}$ boxes. In fact his result is uniform in this regime of λ. An analogous result for $U(n)$ (with slightly worse constant) was obtained recently with de la Salle [MdlS24]. Cassidy's result for $|\lambda|$ up to and including n would answer the above questions about Cayley graphs of S_n.

The fact that not all RAAGs are P**Perm**F does not have a direct implication to e.g. closed hyperbolic 3 manifold groups in general because when one induces a permutation representation from an infinite index subgroup of a RAAG the resulting permutation representation is of an infinite set and so cannot be used to potentially prove P**Perm**F for the RAAG.

2.4 Non-examples

We discuss here f.g. residually finite groups[5]. With M. de la Salle we established [MdlS23] that $\mathrm{SL}_d(\mathbf{Z})$ is not P**Mat**F for $d \geqslant 4$. This leaves a curious gap at $d = 3$. The reason for this gap is that we rely on the following fact, established in *(ibid.):* Every non-trivial finite dimensional unitary representation of $\mathrm{SL}_4(\mathbf{Z})$ has a non-zero $\mathrm{SL}_2(\mathbf{Z})$-invariant vector. This in turn means that the action of $\mathrm{SL}_2(\mathbf{Z})$ in this representation has no spectral gap, and so when restricted to $\mathrm{SL}_2(\mathbf{Z})$, a putative sequence of representations of $\mathrm{SL}_4(\mathbf{Z})$ that strongly converge to the regular representation, cannot converge to the regular representation of $\mathrm{SL}_2(\mathbf{Z})$ — which does have a spectral gap. In light of Lemma 2.1 this is a contradiction.

However, there are f.d. irreducible unitary representations of $\mathrm{SL}_3(\mathbf{Z})$ with dimension tending to infinity and without non-zero $\mathrm{SL}_2(\mathbf{Z})$-invariant vectors. See *(ibid.)* for details — this example is due to Deligne.

2.5 Connection to the Fell topology on the unitary dual

Suppose G is a locally compact topological group. We assume general familiarity with the Fell topology on the unitary dual of G (the equivalence

[5]Non f.g. but residually linear groups have not really been considered in the context of P**Mat**F to the author's knowledge and this might be an interesting thing to investigate further.

classes of continuous unitary representations of G) and also the notion of weak containment of representations[6].

Proposition 2.9. *Suppose that G is any locally compact group. Suppose that $\pi_i : G \to \mathcal{U}(\mathcal{H}_i)$ are any sequence of representations of Γ that strongly convergence to a unitary representation π_∞ of G in the sense that for all $f \in C_c(G)$,*

$$\lim_{i\to\infty} \|\pi_i(f)\| = \|\pi_\infty(f)\|.$$

Then for any compact subset $K \subset \hat{G}\backslash\mathrm{support}(\pi_\infty)$, for $i > i_0(K)$, no element of K is weakly contained in π_i.

Remark 2.10. To attempt to promote Proposition 2.9 to an if and only if statement in the case $\pi_\infty = \lambda_G$, one needs to (at least) also discuss possible discrete series representations of G (for example, in the case $G = \mathrm{PSL}_2(\mathbb{R})$ this issue is already present). The point is that a diagonal matrix coefficient of an integrable discrete series[7] will act as a non-zero projection in $\pi_\infty = \lambda_G$ and therefore needs to act in a non-zero way in any π_i when i is sufficiently large.

Proof. One can show directly from definition of Fell topology that for any $f \in C_c(G)$, the map

$$\pi \mapsto \|\pi(f)\| \tag{2.2}$$

is continuous in the Fell topology. For completeness we give this argument. Given $\pi \in \hat{G}$, suppose that ξ is such that $\|\xi\| = 1$ and

$$\langle \pi(f^* * f)\xi, \xi\rangle = \langle \pi(f)\xi, \pi(f)\xi\rangle > \|\pi(f)\|^2 - \epsilon.$$

Essentially by definition of Fell topology (see [BdlHV08, Prop. F.2.4]) there is open set around π consisting of π' such that

$$|\langle \pi'(g)\xi', \xi'\rangle - \langle \pi(g)\xi, \xi\rangle| < \frac{\epsilon}{1 + \|f^* * f\|_{L^1}}$$

[6] In the Princeton seminar I proved Proposition 2.9 in the case the compact set K was a point. M. de la Salle pointed out the extension to compacts.

[7] Integrable meaning having matrix coefficients in L^1; not all discrete series have this property.

for some ξ' and for all g in the support of $f^* * f$. This gives $\|\xi'\|^2 > 1 - \epsilon$ (take $g = 1$) and integrating with $f^* * f$ weights

$$\begin{aligned}\|\pi'(f)\|^2(1-\epsilon) &> \|\pi'(f)\|^2\|\xi'\|^2 \geqslant \langle \pi'(f^* * f)\xi', \xi'\rangle > \langle \pi(f^* * f)\xi, \xi\rangle - \epsilon \\ &> \|\pi(f)\|^2 - 2\epsilon\end{aligned}$$

for π' in this open set. This proves continuity of (2.2).

Now let K be as in the statement of the proposition. For every $\pi \in K$, as π is not in the support of π_∞ there exists $f_\pi \in C_c(G)$ and $\eta_\pi > 1$ such that $\|\pi(f_\pi)\| > \eta_\pi \|\pi_\infty(f_\pi)\|$. By the previous assertion there is on open neighborhood W_π of π where this inequality still holds. By compactness of K we then obtain a finite list of functions $f_1, \ldots, f_r \in C_c(G)$ and $\eta > 1$ such that for all $\pi \in K$,

$$\|\pi(f_j)\| > \eta \|\pi_\infty(f_j)\|$$

for some f_j. But for large enough $i > i_0$, from strong convergence

$$\|\pi_i(f_j)\| < \eta \|\pi_\infty(f_j)\|$$

for all j. Combining the above two inequalities gives for some f_j, $\|\pi_i(f)\| < \|\pi(f_j)\|$ so π is not in the support of π_i. □

3 Applications

3.1 An induction principle

The following theorem is at the heart of applications of strong convergence to spectral geometry.

Theorem 3.1. *Suppose G is locally compact and Γ is a cocompact lattice in G. If $\rho_i \xrightarrow{\text{strong}} \rho_\infty$ then*

$$\mathrm{Ind}_\Gamma^G \rho_i \xrightarrow{\text{strong}} \mathrm{Ind}_\Gamma^G \rho_\infty.$$

In applications usually one wants $\rho_\infty = \lambda_\Gamma$ so that $\mathrm{Ind}_\Gamma^G \rho_i \xrightarrow{\text{strong}} \mathrm{Ind}_\Gamma^G \lambda_\Gamma = \lambda_G$. In light of Proposition 2.9, and in the case G is semisimple Lie, the identification of the unitary dual with spectral parameters, it yields a type of spectral convergence of $\mathrm{Ind}_\Gamma^G \rho_i$ to the Plancherel measure of G.

The downside of this general argument is that

a. it does not give an effective rate of convergence of spectral parameters,

b. it does not apply as-is to non-uniform lattices.

Both these issues have been dealt with in special instances (see [MT24] and [HM23] regarding Point **a)** and [HM23] regarding Point **b**)).

We now address the proof of Theorem 3.1. It relies on the following type of 'matrix amplification' that is well-known in the literature e.g. [HT05, §9]. We take the chance to record an effective version of this lemma provided by Mikael de la Salle.

Lemma 3.2 (Effective matrix amplication)**.** *Let A be a C^*-algebra and $x \in M_n(A)$. For every integer p*

$$\|x\|_{M_n(A)} \in [1, n^{\frac{1}{2p}}] \max_i \| ((x^*x)^p)_{i,i} \|_A^{\frac{1}{2p}}.$$

As a result,

Proposition 3.3. *If Γ is discrete and $\{\rho_i\}_{i=1}^{\infty}$ are a sequence of unitary representations of Γ with $\rho_i \xrightarrow{\text{strong}} \lambda_\Gamma$ then for all $r \in \mathbb{N}$ and all $z \in \mathbf{Mat}_{r\times r}(\mathbb{C}) \otimes_{\mathbb{C}} \mathbb{C}[\Gamma]$,*

$$\|[\mathrm{id} \otimes \rho_i](z)\| \to \|\mathrm{id} \otimes \lambda_\Gamma(z)\|.$$

Proof of Theorem 3.1. Given $f \in C_c(G)$, $\pi(f)$ acts on $L^2(G, \rho)$ by

$$\pi(f)[\phi](h) = \int_G f(g)\phi(g^{-1}h)d\mu(g) = \int_G f(hg)\phi(g^{-1})d\mu(g).$$

where μ is the left invariant Haar measure. This is the same as

$$\begin{aligned}\sum_{\gamma\in\Gamma} \int_F f(h\gamma g)\phi(g^{-1}\gamma^{-1})d\mu(g) &= \sum_{\gamma\in\Gamma} \int_F f(h\gamma g)\rho(\gamma^{-1})\phi(g^{-1})d\mu(g) \\ &= \sum_{\gamma} \rho(\gamma^{-1})[A_f(\gamma)\phi|_F]\end{aligned}$$

where $A_f(\gamma) : L^2(F, V) \to L^2(F, V)$ such that if $\psi \in L^2(F, V)$ and $h \in F$,

$$A_f(\gamma)[\psi](h) \stackrel{\text{def}}{=} \int_F f(h\gamma g)\psi(g^{-1}). \tag{3.1}$$

Using $L^2(F,V) \cong L^2(F) \otimes V$ we obtain a unitary conjugacy

$$\pi(f) \cong \sum_{\gamma \in \Gamma} a_f(\gamma) \otimes \rho(\gamma^{-1})$$

where $a_f(\gamma)$ is defined by the same formula (3.1) as A_f but acting on $L^2(F)$.

Now we make two observations. Firstly, $a_f(\gamma) = 0$ unless there are h and g in F such that $h\gamma g \in \mathrm{supp}(f)$. Let $K \subset G$ be compact such that

$$G = \bigcup_{\gamma \in \Gamma} \gamma C$$

and $F \subset C$. The above event is majorized by $\gamma \subset C^{-1}\mathrm{supp}(f)C^{-1}$ which is contained in a compact set. This compact subset can meet only finitely many elements of γ. So

$$\gamma \mapsto a_f(\gamma)$$

has finite support.

Secondly, $a_f(\gamma)$ is an integral operator with kernel $K_\gamma(h,g) = f(h\gamma g^{-1})$. Since f is continuous, and the closure of F is compact, it is bounded on F and even Hilbert-Schmidt.

So the image of this conjugacy is contained in

$$\mathbb{C}[\Gamma] \otimes_{\mathbb{C}} HS(F)$$

where $HS(F)$ are the Hilbert-Schmidt operators on $L^2(F)$ and hence generate a C^*-subalgebra of $C^*_{\mathrm{red}}(\Gamma) \otimes \mathcal{K}(L^2(F))$ where $\mathcal{K}$ are the compact operators on a separable Hilbert space.

Now by Proposition 3.3 together with approximation of compact operators by finite rank ones, we have for all $f \in C_c(G)$ and notation as above, if $\pi_i = \mathrm{Ind}_\Gamma^G \rho_i$ and $\rho_i \xrightarrow{\text{strong}} \rho_\infty$ then

$$\|\pi_i(f)\| = \|\sum_{\gamma \in \Gamma} a_f(\gamma) \otimes \rho_i(\gamma^{-1})\| \to \|\sum_{\gamma \in \Gamma} a_f(\gamma) \otimes \lambda_\Gamma(\gamma^{-1})\| \cong \|\mathrm{Ind}_\Gamma^G \rho_\infty\|.$$

This proves Theorem 3.1. $\square$

3.2 Hyperbolic surfaces

In this section Γ is the fundamental group of a closed orientable surface of genus $g \geqslant 2$. Moreover, we discretely embed $\Gamma \hookrightarrow \mathrm{PSL}_2(\mathbb{R})$ in some fixed but arbitrary way, fixing a hyperbolic structure on a genus g surface $X \stackrel{\text{def}}{=} \Gamma\backslash\mathbb{H}$. Importantly, this embedding could be arithmetic.

By Corollary 2.5, there is a sequence $\{\phi_i \in \mathrm{Hom}(\Gamma, S_{n_i})\}_{i=1}^{\infty}$ such that the indcued $\rho_i = \mathrm{std} \circ \phi_i$ satsify $\rho_i \xrightarrow{\text{strong}} \lambda_\Gamma$. Hence

$$\mathrm{Ind}_\Gamma^G \rho_i \xrightarrow{\text{strong}} \lambda_{\mathrm{PSL}_2(\mathbb{R})}$$

by Theorem 3.1.

Proceeding depends on knowing the unitary dual of $\mathrm{PSL}_2(\mathbb{R})$ and the Plancherel measure. Of interest here are the complementary series which are outside the support of the Plancherel measure. Now Proposition 2.9 implies that for any compact subset K of the complementary series, for $i \gg_K 1$ no member of K is weakly contained in

$$\mathrm{Ind}_\Gamma^G \rho_i.$$

A note. (On fibered products)

We now make one more observation. Because ρ_i is derived from ϕ_i, the space of $\mathrm{Ind}_\Gamma^G \rho_i$ is same as L^2 sections of the fibered product

$$\Gamma\backslash_{\phi_i}(G \times \ell_0^2([n_i])).$$

In turn, such sections are the same as L^2 functions on

$$\Gamma\backslash_{\phi_i}(G \times [n_i])$$

that have mean zero in every fiber (the above is a covering space of $\Gamma\backslash G$).

By the relation between the Casimir operator of $\mathrm{PSL}_2(\mathbb{R})$ and the Laplacian on the hyperbolic surface

$$X_{\phi_i} \stackrel{\text{def}}{=} \Gamma\backslash_{\phi_i}(\mathbb{H} \times [n_i]) = \Gamma\backslash_{\phi_i}(G \times [n_i])/P\mathrm{SO}(2),$$

one obtains as conclusion:

Theorem 3.4. *As $i \to \infty$,*

$$\mathrm{spec}(\Delta_{X_{\phi_i}}) \cap \left[0, \frac{1}{4} - o(1)\right] = \mathrm{spec}(\Delta_X) \cap \left[0, \frac{1}{4} - o(1)\right].$$

By choosing X so that $\Delta_X$has no eigenvalues below $\frac{1}{4}$, one obtains a sequence of closed hyperbolic surfaces (covering X) with genus tending to ∞ and first non-zero eigenvalue tending to $\frac{1}{4}$. [8]

Remark 3.5. This result, which established a conjecture of Buser [Bus78] was first obtained in joint work of the author with Hide [HM23] by a related method. At this time, we only had P**Perm**F for free groups, so we worked with non-compact surfaces with free fundamental groups and compactified at the end of the argument following Buser–Burger–Dodziuk [BBD88]. The problem this introduced was that we did not have access to Theorem 3.1 so we had to make a more involved argument using the resolvent of the Laplacian and cusp-patching techniques. This technique also yields the following theorem, taking Bordenave–Collins as input.

Theorem 3.6 (Hide–Magee)**.** *Let X be a finite-area non-compact hyperbolic surfaces so that $\pi_1(X) \cong \mathbf{F}$ for some free group $\mathbf{F}$. Let ϕ_n now be a uniform random element of $\mathrm{Hom}(\mathbf{F}, S_n)$. Then X_{ϕ_n} is a uniform random degree$-n$ covering space of X. With probability tending to one as $n \to \infty$*

$$\mathrm{spec}(\Delta_{X_{\phi_n}}) \cap \left[0, \frac{1}{4} - o(1)\right] = \mathrm{spec}(\Delta_X) \cap \left[0, \frac{1}{4} - o(1)\right].$$

This theorem forms a part of much recent activity on the spectral gaps of random hyperbolic surfaces [MN20, MP23, MNP22, WX22, LW24, AM23, Hid23, HT24, AM24].

It is an interesting question to what extent the 'induction principle' obtained above in Theorem 3.1 can be extended to general non-cocompact and even infinite covolume lattices in e.g. reductive groups. In recent work [CMN24], it has been shown that induction of strong convergence works well in the setting of conformally compact hyperbolic surfaces (of infinite area), and even gives resonance free regions — a phenomenon that cannot be seen solely in the representation theory of $\mathrm{Ind}_{\Gamma}^{\mathrm{PSL}_2(\mathbb{R})}\rho_i$. These questions should be pursued in future work.

[8] In fact, one can arrange so that the original X is arithmetic and so obtain the above conclusion where all surfaces are arithmetic. Elaboration on this (also using strong convergence as an essential ingredient) one can prove that <u>every</u> $x \in [0, \frac{1}{4}]$ is a limit point of λ_1 of arithmetic hyperbolic surfaces [Mag24].

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